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Uniform asymptotic and JWKB expansions for anharmonic oscillators

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Abstract. We show explicitly the relation between the uniform asymptotic and the Jeffreys–Wentzel–Kramers–Brillouin (JWKB) wavefunctions, and between the matching of uniform asymptotic expansions and the complete JWKB connection formulae written in terms of Stokes multipliers and loop integrals. As an application we give a unified derivation of the asymptotic behaviour of the imaginary part of the resonances in anharmonic oscillators and, via dispersion relations, the corresponding asymptotic behaviour of the Rayleigh–Schrödinger perturbation theory coefficients.

1. Introduction

Matching of uniform asymptotic expansions is one of the standard methods to derive Jeffreys–Wentzel–Kramers–Brillouin (JWKB) connection formulae [1]. In its simplest version, the uniform asymptotic expansions around turning points are interpreted in the sense of Poincaré, in which there is no analytic criterion to retain subdominant terms, although it may be numerically convenient to do so (see, for example, pp 76–78 in [2]). The resulting connection formulae have been discussed extensively by Olver [2] and Fröman and Fröman [3], who derived rigorous bounds for the truncation error and studied several examples and generalizations.

Silverstone [4], however, supplemented the matching method with the Borel summability of the underlying asymptotic expansions of confluent hypergeometric functions [5], and obtained 'complete' connection formulae in which the correct subdominant terms are identified and interpreted unambiguously. That is to say, the Borel-summable asymptotic solutions of the Schrödinger equation are encodings of the same exact solution in different regions of the complex plane separated by Stokes lines, and the complete connection formulae specify the relation between these expansions. For numerical purposes, a finite number of terms of the expansion in any particular region are used to calculate an approximation to the Borel sum via Padé approximants or conformal transformations, not by partial summation. (Classically forbidden regions are instances of Stokes lines, where the Borel-summable asymptotic expansions may change discontinuously; the evaluation of the function on a Stokes line has to be done by continuity from the appropriate Borel-summable expansions on either side of the line.)

In fact, following an idea of Balian and Bloch [6], Voros [7] studied the analytic structure of the Borel transform with respect to $\hbar^{-1}$ of the JWKB wavefunctions, arriving at an equivalent form of the complete connection formulae. This procedure was later reformulated in geometrical terms, extended and applied to several examples by Delabaere *et al* [8–11].

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Our purpose in this paper is to show explicitly the relation between the uniform and JWKB wavefunctions, and between the matching of expansions and the connection formulae in terms of Stokes multipliers and JWKB loop integrals (analytic Voros symbols, in the terminology of [8–11]) for the Schrödinger equation for perturbed harmonic oscillators

$$-\tfrac{1}{2}\psi''(x) + (\tfrac{1}{2}x^2 - gx^p - E)\psi(x) = 0 \qquad (1)$$

where $p > 2$ is an integer. If p is even, we look for even and odd purely outgoing (or ingoing) solutions both at plus and minus infinity; if p is odd, we look for solutions that are exponentially decreasing at minus infinity and purely outgoing (or ingoing) at plus infinity. (Due to these different boundary conditions and the corresponding Stokes graphs, we will discuss first the case of p odd, and point out later the modifications required for p even.) We mention that in the cases of $p = 3$ and 4 there are explicit expressions of the Stokes multipliers [12] as convergent series whose terms are functions of the coefficients of the polynomial potential (g and E in our notation), but these series are very slowly convergent and not well suited to solve the stated eigenvalue problem, which requires, as we do in this paper, more sophisticated methods ultimately based on comparison with the Weber and Airy equations.

As a simple consequence of our results we also obtain a unified derivation of the asymptotic behaviour of the imaginary part of the resonances as $g \to 0$ and, via dispersion relations, the asymptotic behaviour of the Rayleigh–Schrödinger perturbation theory (RSPT) coefficients for these anharmonic oscillators. This general equation includes as particular instances the recent results of Bender and Dunne [13] for the cubic anharmonic oscillator and the results for quartic, sextic and octic oscillators quoted by Ivanov [14, 15].

The layout of the paper is as follows: in sections 2 and 3 we build uniform expansions around the origin and around the outer turning point respectively; in section 4 we discuss the matching of these asymptotic expansions, the leading-order solution to the matching condition, and the imaginary part of the resonances; in section 5 we discuss the modifications required for p even; section 6 is devoted to clarify the relation between the uniform asymptotic and the JWKB expansions, and the paper ends with a brief summary.

2. Uniform expansion around the origin for p odd

We do not work directly with equation (1), but scale the independent variable by

$$x = h^{-1/2}z \qquad (2)$$

where

$$g = \tfrac{1}{2}h^{p/2-1}. \qquad (3)$$

Thus we get a scaled Schrödinger equation

$$-h^2\psi''(z) + (z^2 - z^p - 2hE)\psi(z) = 0 \qquad (4)$$

with an unperturbed double turning point at the origin and an unperturbed simple turning point fixed at $z = 1$ (and a symmetric partner at $z = -1$ if p is even).

We build a uniform asymptotic expansion to $\psi(z)$ around the double-turning point at the origin following the method of Langer [16], Cherry [17], Lynn and Keller [18], and Silverstone *et al* [19]. The comparison equation will be Weber's differential equation [20], and the solution with appropriate asymptotic behaviour (see section 4) is the parabolic cylinder function $D_{\nu-1/2}(z)$. Therefore we write

$$\psi(z) = [u'(z)]^{-1/2} D_{\nu-1/2}\left[-\left(\frac{2}{h}\right)^{1/2} u(z)\right] \qquad (5)$$

which gives the following equation for $u(z)$:

$$u(z)^2 u'(z)^2 = z^2 - z^p - 2h[E - \nu u'(z)^2] + \frac{h^2}{2}\{u, z\} \tag{6}$$

where $\{u, z\}$ is the Schwarzian derivative,

$$\{u, z\} = \frac{u'''(z)}{u'(z)} - \frac{3}{2}\left(\frac{u''(z)}{u'(z)}\right)^2. \tag{7}$$

Next we substitute the asymptotic expansions

$$u(z) = \sum_{k=0}^{\infty} u_k(z) h^k \tag{8}$$

$$E = \sum_{k=0}^{\infty} E^{(k)}(\nu)(\tfrac{1}{2}h^{p/2-1})^k \tag{9}$$

into equation (6) and equate powers of h to obtain a system of differential equations whose first two members are

$$u_0(z)u_0'(z) = (z^2 - z^p)^{1/2} \tag{10}$$

$$u_0(z)u_1'(z) + u_0'(z)u_1(z) = -\frac{E^{(0)}(\nu)}{u_0(z)u_0'(z)} + \nu\frac{u_0'(z)}{u_0(z)}. \tag{11}$$

We integrate these equations recursively and fix the coefficients $E^{(k)}(\nu)$ by the requirement that $u_k(z)$ be regular at the origin [17]. Equation (10) can be integrated in terms of the Gauss hypergeometric function [20]

$$u_0(z)^2 = 2\int_0^z (s^2 - s^p)^{1/2}\,\mathrm{d}s = z^2 F(-\tfrac{1}{2}, q; q+1; z^{2/q}) \tag{12}$$

where we have defined

$$q = \frac{2}{(p-2)}. \tag{13}$$

To integrate equation (11) we note that

$$\int \frac{\mathrm{d}s}{(s^2 - s^p)^{1/2}} = \frac{q}{2}\ln\frac{1 - (1 - s^{2/q})^{1/2}}{1 + (1 - s^{2/q})^{1/2}} \tag{14}$$

$$\sim -q\ln 2 + \ln s + \frac{q}{4}s^{2/q} + \cdots. \tag{15}$$

Therefore, we avoid the logarithmic singularity in $u_1(z)$ by setting

$$E^{(0)}(\nu) = \nu \tag{16}$$

and get

$$u_0(z)u_1(z) = \frac{\nu}{2}\ln[u_0(z)^2] - \frac{q\nu}{2}\ln\frac{1 - (1 - z^{2/q})^{1/2}}{1 + (1 - z^{2/q})^{1/2}} - q\nu\ln 2. \tag{17}$$

Carrying on this procedure, we find out that the coefficients $E^{(k)}(\nu)$ are polynomials in ν with rational coefficients, the first few of which are shown in table 1. We advance that these polynomials have precisely the form of the RSPT coefficients when expressed as polynomials in $n + \frac{1}{2}$, except that they are now functions of the as yet unspecified parameter ν. An elementary parity argument shows that for p odd the odd RSPT coefficients $E^{(2k+1)}(\nu)$ vanish, but we keep the generic notation (9) that can also be used for p even. We also mention that Caliceti [21] has recently proved rigorously that, in appropriate regions of the coupling constant plane, these RSPT series are Borel summable to the resonances.

Table 1. Lowest RSPT coefficients $E^{(k)}(\nu)$ as polynomials in ν.

p	$-E^{(1)}(\nu)$	$-E^{(2)}(\nu)$	$-E^{(3)}(\nu)$	$-E^{(4)}(\nu)$
3	0	$\frac{7}{16}+\frac{15}{4}\nu^2$	0	$\frac{1155}{64}\nu+\frac{705}{16}\nu^3$
4	$\frac{3}{8}+\frac{3}{2}\nu^2$	$\frac{67}{16}\nu+\frac{17}{4}\nu^3$	$\frac{1539}{256}+\frac{1707}{32}\nu^2+\frac{375}{16}\nu^4$	$\frac{305141}{1024}\nu+\frac{89165}{128}\nu^3+\frac{10689}{64}\nu^5$
5	0	$\frac{1107}{256}+\frac{1085}{32}\nu^2+\frac{315}{16}\nu^4$	0	$\frac{115763715}{8192}\nu+\frac{90794795}{2048}\nu^3+\frac{13519905}{512}\nu^5+\frac{494385}{128}\nu^7$
6	$\frac{25}{8}\nu+\frac{5}{2}\nu^3$	$\frac{19277}{256}\nu+\frac{4145}{32}\nu^3+\frac{393}{16}\nu^5$	$\frac{11719955}{2048}\nu+\frac{7364155}{512}\nu^3+\frac{735945}{128}\nu^5+\frac{14745}{32}\nu^7$	$\frac{224719341733}{262144}\nu+\frac{44111182385}{16384}\nu^3+\frac{12791269491}{8192}\nu^5+\frac{264832005}{1024}\nu^7+\frac{11451165}{1024}\nu^9$

3. Uniform expansion around the outer turning point

The uniform expansion around the outer turning point at $z = 1$ is built following the same steps, except that the appropriate comparison equation is Airy's differential equation [20], and the solution with appropriate asymptotic behaviour (see again section 4) is the Airy function $\mathrm{Ai}^{(\pm)}(z) = \mathrm{Bi}(z) \pm \mathrm{i}\,\mathrm{Ai}(z)$. Therefore we set

$$\psi(z) = [v'(z)]^{-1/2}\mathrm{Ai}^{(\pm)}[h^{-2/3}v(z)] \tag{18}$$

into the Schrödinger equation (4), and obtain the equation for $v(z)$,

$$v(z)v'(z)^2 = z^2 - z^p - h2E + \frac{h^2}{2}\{v, z\}. \tag{19}$$

Substitution of the asymptotic series for the wavefunction

$$v(z) = \sum_{k=0}^{\infty} v_k(z)h^k \tag{20}$$

and for the energy (9) into (19) leads to a system whose first two equations are

$$v_0(z)v_0'(z)^2 = z^2 - z^p \tag{21}$$

$$v_1(z)v_0'(z)^2 + 2v_0(z)v_0'(z)v_1'(z) = -2\nu. \tag{22}$$

Again, we integrate these equations recursively, and the same $E^{(k)}(\nu)$ determined in the previous section free the $v_k(z)$ of singularities at $z = 1$. For later reference, we quote the corresponding solutions:

$$v_0(z)^{3/2} = \tfrac{3}{2}\int_z^1 (s^2 - s^p)^{1/2}\,\mathrm{d}s \tag{23}$$

$$= \frac{q}{2}(1 - z^{2/q})^{3/2}F(1 - q, \tfrac{3}{2}; \tfrac{5}{2}; 1 - z^{2/q}) \tag{24}$$

$$= \tfrac{3}{4}qB(q, \tfrac{3}{2}) - \tfrac{3}{4}z^2F(-\tfrac{1}{2}, q; 1 + q; z^{2/q}) \tag{25}$$

$$v_0(z)^{1/2}v_1(z) = \frac{\nu q}{2}\ln\frac{1 - (1 - z^{2/q})^{1/2}}{1 + (1 - z^{2/q})^{1/2}} \tag{26}$$

where $B(q, \tfrac{3}{2})$ in equation (25) denotes Euler's beta function.

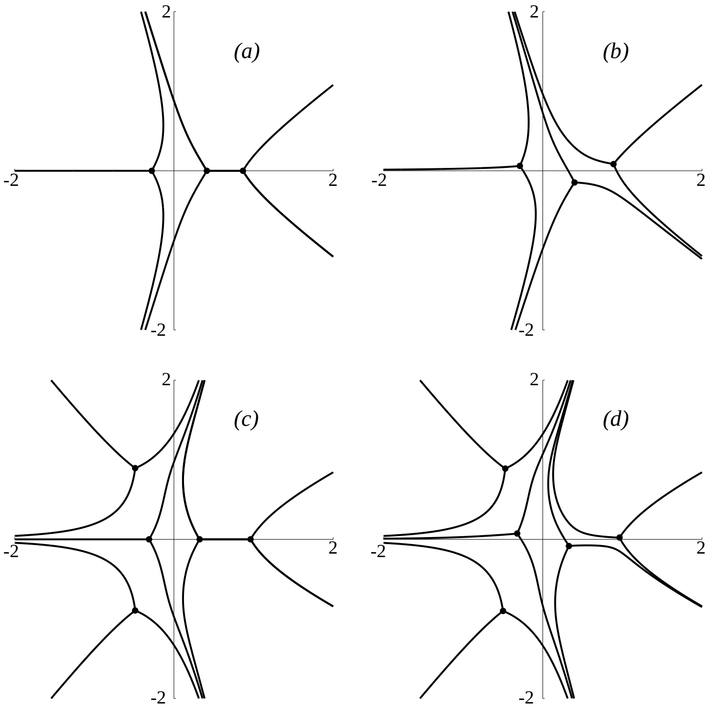

Figure 1. Stokes graphs for the scaled Schrödinger equation (4) with: (*a*) $p = 3$ and $2hE = 1/10$; (*b*) $p = 3$ and $2hE = 1/10 - \mathrm{i}/20$; (*c*) $p = 5$ and $2hE = 1/10$; and (*d*) $p = 5$ and $2hE = 1/10 - \mathrm{i}/20$.

4. Direct matching of the asymptotic expansions

4.1. Matching region

In figures 1(*a*) and (*c*) we show the Stokes graphs for equation (4) with real $2hE = \frac{1}{10}$ and $p = 3$ and 5, where the 'under the barrier' classically forbidden matching region is a finite Stokes line. These patterns are resolved into Stokes graphs with only infinite Stokes lines as soon as the constant term $2hE$ acquires a nonzero imaginary part—we illustrate this case in figures 1(*b*) and (*d*), where $2hE = \frac{1}{10} - \frac{\mathrm{i}}{20}$ and the real axis can be taken as the matching path. (The imaginary part in the eigenvalue problem is in fact exponentially small as $h \to 0$; the value of $\mathrm{Im}\,(2hE)$ in the figure is sufficiently large to show clearly the splitting but sufficiently small to avoid the first change in the topology of the Stokes graphs.) We will match the Borel-summable asymptotic expansions of the parabolic cylinder-based and Airy-based wavefunctions following this point of view. The required asymptotic expansions can be easily derived from the general results for confluent hypergeometric functions in [5], and can be conveniently written in terms of the formal generalized hypergeometric series

$$ {}_2F_0(a_1, a_2; ; z) = \sum_{k=0}^{\infty} (a_1)_k (a_2)_k \frac{z^k}{k!} \tag{27} $$

where $(a)_k$ denotes the Pochhammer symbol: $(a)_0 = 1$, $(a)_k = a(a+1)\cdots(a+k-1)$.

4.2. Asymptotic expansions at the origin

The Borel-summable asymptotic expansions for the parabolic cylinder functions in the relevant sectors of the complex plane are

$$D_{\nu-1/2}(z) \sim z^{\nu-1/2} e^{-z^2/4} {}_2F_0(\tfrac{1}{4}-\tfrac{\nu}{2}, \tfrac{3}{4}-\tfrac{\nu}{2};\,;-2z^{-2}) \qquad (-\tfrac{1}{2}\pi < \arg z < \tfrac{1}{2}\pi) \tag{28}$$

$$D_{\nu-1/2}(z e^{\mp i\pi}) \sim \frac{(2\pi)^{1/2}}{\Gamma(\frac{1}{2}-\nu)} z^{-\nu-1/2} e^{z^2/4} {}_2F_0(\tfrac{1}{4}+\tfrac{\nu}{2}, \tfrac{3}{4}+\tfrac{\nu}{2};\,;2z^{-2}) \pm i e^{\mp i\nu\pi} z^{\nu-1/2} e^{-z^2/4} {}_2F_0(\tfrac{1}{4}-\tfrac{\nu}{2}, \tfrac{3}{4}-\tfrac{\nu}{2};\,;-2z^{-2}) \quad (0 < \pm \arg z < \tfrac{1}{2}\pi). \tag{29}$$

As we anticipated in section 2, equation (28) shows that the uniform wavefunction (5) is exponentially decreasing along the negative real axis, while in the matching region $u(z) \sim z > 0$ with $\mathrm{sgn}[\mathrm{Im}\,(-(2/h)^{1/2}u(z))] = \mathrm{sgn}[\mathrm{Im}\,h]$, and equation (29) shows that (5) is a linear combination of an exponentially increasing term and an exponentially decreasing (subdominant) term uniquely defined via Borel summability.

4.3. Asymptotic expansions at the outer turning point

Similarly, the Borel-summable asymptotic expansions for the Airy functions in the relevant sectors are

$$\mathrm{Ai}^{(\pm)}(z) \sim \pi^{-1/2} z^{-1/4} e^{\frac{2}{3}z^{3/2}} {}_2F_0(\tfrac{1}{6}, \tfrac{5}{6};\,;\tfrac{3}{4}z^{-3/2}) \qquad (-\tfrac{4}{3}\pi < \pm\arg z < 0) \tag{30}$$

$$\mathrm{Ai}^{(\pm)}(z) \sim \pi^{-1/2} z^{-1/4} [e^{\frac{2}{3}z^{3/2}} {}_2F_0(\tfrac{1}{6}, \tfrac{5}{6};\,;\tfrac{3}{4}z^{-3/2}) \pm i e^{-\frac{2}{3}z^{3/2}} {}_2F_0(\tfrac{1}{6}, \tfrac{5}{6};\,;-\tfrac{3}{4}z^{-3/2})] \qquad (0 < \pm\arg z < \tfrac{2}{3}\pi). \tag{31}$$

As we anticipated in section 3, equation (30) shows that the uniform wavefunction (18) represents a purely outgoing wave along (slightly above or below) the positive real axis.

4.4. Formal matching

First note that for $-q\pi < \arg h < 0$ the matching is trivial because the first term in equation (29) must vanish, i.e. the argument of the gamma function must be zero or a negative integer, and we obtain

$$\nu = n + \tfrac{1}{2} \qquad (n = 0, 1, 2, \ldots). \tag{32}$$

For sufficiently small $\arg h > 0$ we implement the matching by equating the ratios of the dominant to the subdominant terms in the two uniform asymptotic expansions (29) and (31) valid in the matching region [19], i.e. we get the index ν as the solution of

$$\frac{\Gamma(\frac{1}{2}-\nu)}{(2\pi)^{1/2}e^{i(\nu+1)\pi}} \left[\frac{2u(z)^2}{h}\right]^{\nu} \exp\left[-\frac{u(z)^2}{h}\right] \frac{{}_2F_0(\frac{1}{4}-\frac{\nu}{2}, \frac{3}{4}-\frac{\nu}{2};\,;-hu(z)^{-2})}{{}_2F_0(\frac{1}{4}+\frac{\nu}{2}, \frac{3}{4}+\frac{\nu}{2};\,;hu(z)^{-2})} = \exp\left[\frac{4v(z)^{3/2}}{3h}\right] \frac{{}_2F_0(\frac{1}{6}, \frac{5}{6};\,;\frac{3}{4}hv(z)^{-3/2})}{{}_2F_0(\frac{1}{6}, \frac{5}{6};\,;-\frac{3}{4}hv(z)^{-3/2})}. \tag{33}$$

To put this equation in a more convenient form, we use the gamma function reflection formula

$$\Gamma(\tfrac{1}{2}+\nu)\Gamma(\tfrac{1}{2}-\nu) = \frac{\pi}{\cos(\pi\nu)} \tag{34}$$

and define a new function

$$f(\nu) = \frac{(2\pi)^{1/2}}{\Gamma(\nu+\frac{1}{2})} \exp\left[-\frac{u(z)^2}{h} + \nu \ln\left(\frac{2u(z)^2}{h}\right) + \ln \frac{{}_2F_0(\frac{1}{4}-\frac{\nu}{2}, \frac{3}{4}-\frac{\nu}{2};\,; -hu(z)^{-2})}{{}_2F_0(\frac{1}{4}+\frac{\nu}{2}, \frac{3}{4}+\frac{\nu}{2};\,; hu(z)^{-2})} \right.$$
$$\left. -\frac{4v(z)^{3/2}}{3h} + \ln \frac{{}_2F_0(\frac{1}{6}, \frac{5}{6};\,; -\frac{3}{4}hv(z)^{-3/2})}{{}_2F_0(\frac{1}{6}, \frac{5}{6};\,; \frac{3}{4}hv(z)^{-3/2})} \right]. \qquad (35)$$

Note that after substitution of the expansions (8) and (20) into equation (35), all the z-dependent terms in the matching process cancel, leaving only the dependence in ν (and, of course, in q and h), and the general odd p matching condition (33) can be written as follows:

$$f(\nu) = e^{i\pi}(e^{i2\pi\nu} + 1). \qquad (36)$$

4.5. *Leading behaviour and lowest-order solution*

Substituting equations (12), (17), (25) and (26) into equation (35) and pulling out the leading-order terms, we find that

$$f(\nu) = \frac{(2\pi)^{1/2} 2^{2q\nu}}{\Gamma(\nu+\frac{1}{2})} \left(\frac{2}{h}\right)^{\nu} \exp\left[-\frac{qB(q,\frac{3}{2})}{h} - \sum_{k=1}^{\infty} f^{(k)}(\nu) h^k \right] \qquad (37)$$

or as a function of the unscaled coupling constant g,

$$f(\nu) = \frac{(2\pi)^{1/2} 2^{\nu}}{\Gamma(\nu+\frac{1}{2})} \left(\frac{2}{g}\right)^{q\nu} \exp\left[-\frac{qB(q,\frac{3}{2})}{(2g)^q} - \sum_{k=1}^{\infty} c^{(k)}(\nu) g^k \right] \qquad (38)$$

where the $c^{(k)}(\nu)$ are polynomials in ν determined by the higher terms in the matching condition. These polynomials can be calculated explicitly for any value of p, and the first few are shown in table 2. Furthermore, equation (37) shows that $f(\nu)$ is an exponentially small function of h, and the matching condition (36) can be put in a form suitable for iterative solution. We set

$$\nu = n + \tfrac{1}{2} + \Delta\nu \qquad (n = 0, 1, 2, \ldots) \qquad (39)$$

into equation (36) to get the equivalent equation

$$\Delta\nu = \frac{1}{2\pi i} \ln\left[1 + f\left(n + \frac{1}{2} + \Delta\nu\right)\right]. \qquad (40)$$

Keeping only the first term in the Taylor expansion of the logarithm, we find that the lowest-(first exponentially small) order solution to the matching condition (36) is

$$\Delta\nu \sim \frac{1}{2\pi i} f\left(n + \frac{1}{2}\right). \qquad (41)$$

4.6. *Asymptotic behaviour of the resonances*

To obtain the asymptotic behaviour of the imaginary part of the resonances, we first substitute equation (32) into (9). The result in terms of the unscaled coupling constant g is

$$E(g) \sim \sum_{k=0}^{\infty} E^{(k)}(n + \tfrac{1}{2}) g^k \qquad (42)$$

readily identified as the Borel summable RSPT (note that, with our choice of the Hamiltonian, for g in the negative imaginary axis we have the typical alternating sign pattern of the Borel-summable series with real Borel sum [21]). Similarly, substituting equations (39) and (41) into equation (9) and expanding to first order in $\Delta\nu$ we get

$$E(g) \sim \sum_{k=0}^{\infty} E^{(k)}\left(n + \frac{1}{2}\right) g^k + \Delta\nu \sum_{k=0}^{\infty} \frac{dE^{(k)}}{d\nu}\left(n + \frac{1}{2}\right) g^k + O(\Delta\nu^2). \qquad (43)$$

Table 2. Lowest $c^{(k)}(\nu)$ coefficients in the expression of $f(\nu)$ (equation (38)) as polynomials in ν.

p	$c^{(1)}(\nu)$	$c^{(2)}(\nu)$	$c^{(3)}(\nu)$	$c^{(4)}(\nu)$
3	0	$\frac{77}{32}+\frac{141}{8}\nu^2$	0	$\frac{13937}{128}\nu+\frac{7717}{32}\nu^3$
4	$\frac{67}{48}+\frac{17}{4}\nu^2$	$\frac{569}{32}\nu+\frac{125}{8}\nu^3$	$\frac{305141}{9216}+\frac{89165}{384}\nu^2+\frac{17815}{192}\nu^4$	$\frac{3105983}{2048}\nu+\frac{195755}{64}\nu^3+\frac{87549}{128}\nu^5$
5	0	$\frac{190539}{9216}+\frac{132245}{1152}\nu^2+\frac{10865}{192}\nu^4$	0	$\frac{2476272807}{36864}\nu+\frac{3183085423}{18432}\nu^3+\frac{210012613}{2304}\nu^5+\frac{14154617}{1152}\nu^7$
6	$\frac{221}{24}\nu+\frac{17}{3}\nu^3$	$\frac{2283899}{7680}\nu+\frac{38459}{96}\nu^3+\frac{10727}{160}\nu^5$	$\frac{1642757413}{64512}\nu+\frac{14725045}{288}\nu^3+\frac{3442219}{192}\nu^5+\frac{55747}{42}\nu^7$	$\frac{64522032953459}{15728640}\nu+\frac{3084767116889}{294912}\nu^3+\frac{2619604188383}{491520}\nu^5+\frac{2484506287}{3072}\nu^7+\frac{1217388017}{36864}\nu^9$

Therefore, the discontinuity along the real g axis is

$$\Delta E = 2\mathrm{i}\,\mathrm{Im}\,E(g) \sim -\mathrm{i}\frac{2^n}{\pi^{1/2}n!}\left(\frac{2}{g}\right)^{q(n+\frac{1}{2})}\exp\left[-\frac{qB(q,\frac{3}{2})}{(2g)^q}\right]\sum_{j=0}^{\infty} b^{(j)}\left(n+\frac{1}{2}\right)g^j \tag{44}$$

where $b^{(j)}(n+\frac{1}{2})$ are the coefficients that result form multiplying out the two power series in g whose lowest terms are given in tables 1 and 2:

$$\sum_{j=0}^{\infty} b^{(j)}(n+\tfrac{1}{2})g^j = \exp\left[-\sum_{k=1}^{\infty} c^{(k)}\left(n+\frac{1}{2}\right)g^k\right]\left(\sum_{k=0}^{\infty}\frac{\mathrm{d}E^{(k)}}{\mathrm{d}\nu}\left(n+\frac{1}{2}\right)g^k\right). \tag{45}$$

Furthermore, substitution of the discontinuity in the asymptotic expansion of the eigenvalues given by equation (44) into the dispersion relation in g^2

$$E^{(k)}(n+\tfrac{1}{2}) = \frac{1}{\pi\mathrm{i}}\int_0^{\infty}\Delta E(g)g^{-k-1}\,\mathrm{d}g \qquad (k\text{ even}) \tag{46}$$

yields the large-order asymptotic behaviour of the RSPT coefficients

$$\begin{aligned} E^{(k)}(n+\tfrac{1}{2}) \sim -&\frac{2^{n+k+q(2n+1)}}{q\pi^{3/2}n!}\left[qB\left(q,\frac{3}{2}\right)\right]^{-(n+\frac{1}{2}+\frac{k}{q})} \\ &\times\Gamma\left(n+\frac{1}{2}+\frac{k}{q}\right)\left[\sum_{j=0}^{\infty} b^{(j)}\left(n+\frac{1}{2}\right)\left[qB\left(q,\frac{3}{2}\right)\right]^{j/q} \right.\\ &\left.\times\frac{\Gamma(n+\frac{1}{2}+\frac{k}{q}-\frac{j}{q})}{2^j\Gamma(n+\frac{1}{2}+\frac{k}{q})}\right] \qquad (k\text{ even}). \end{aligned} \tag{47}$$

In fact, by the same parity argument mentioned above, the odd $b^{(2j+1)}(n+\frac{1}{2})$ vanish, but again we keep the general notation to compare later with the analogous equation for even p.

5. Modifications for p even

As we mentioned in the introduction, if p is even we look for purely outgoing (or ingoing) wavefunctions with well-defined parity. In this case, suitable even and odd Langer–Cherry solutions around the origin can be written in terms of the confluent hypergeometric function $F(a; b; z)$ [20]

$$\psi_{\mathrm{even}}(z) = [u'(z)]^{-1/2}\mathrm{e}^{-u(z)^2/(2h)}F(\tfrac{1}{4} - \tfrac{\nu}{2}; \tfrac{1}{2}; h^{-1}u(z)^2) \tag{48}$$

$$\psi_{\mathrm{odd}}(z) = [u'(z)]^{-1/2}u(z)\mathrm{e}^{-u(z)^2/(2h)}F(\tfrac{3}{4} - \tfrac{\nu}{2}; \tfrac{3}{2}; h^{-1}u(z)^2) \tag{49}$$

and these wavefunctions have to be matched to the Airy wavefunction (18) anchored to the outer turning point. We only sketch the main steps of the derivation and give the results, because the procedure is analogous to that in the previous section with two obvious modifications: we have to use the Borel-summable asymptotic expansions for the confluent hypergeometric functions [20], and it is necessary to deal separately with $n = 2k$ and $n = 2k + 1$, although the final results can be stated independently of the parity of n.

For $-2q\pi < \arg h < 0$ the matching is again trivial and yields just $\nu = n + \frac{1}{2}$ and the Borel summable RSPT (42) (note that, with our choice of Hamiltonian, for g negative we have the typical alternating sign pattern of the Borel-summable series with real Borel sum), while for sufficiently small $\arg h > 0$, and defining $f(\nu)$ exactly as in equation (35), the even p matching condition is

$$\Delta\nu = \frac{1}{\pi\mathrm{i}}\ln\left[1 + f\left(n + \frac{1}{2} + \Delta\nu\right)\right] \tag{50}$$

the first exponentially small order solutions is

$$\Delta\nu = \frac{1}{\pi\mathrm{i}}f\left(n + \frac{1}{2}\right) \tag{51}$$

and the discontinuity along the positive real g axis is

$$\Delta E = 2\mathrm{i}\,\mathrm{Im}\,E(g) \sim -\mathrm{i}\frac{2^{n+1}}{\pi^{1/2}n!}\left(\frac{2}{g}\right)^{q(n+\frac{1}{2})}\exp\left[-\frac{qB(q,\frac{3}{2})}{(2g)^q}\right]\sum_{j=0}^{\infty}b^{(j)}\left(n + \frac{1}{2}\right)g^j. \tag{52}$$

Finally, substituting equation (52) into the dispersion relation in g

$$E^{(k)}(n + \tfrac{1}{2}) = \frac{1}{2\pi\mathrm{i}}\int_0^{\infty}\Delta E(g)g^{-k-1}\,\mathrm{d}g \tag{53}$$

yields the large-order asymptotic behaviour of the RSPT coefficients for p even

$$E^{(k)}(n + \tfrac{1}{2}) \sim -\frac{2^{n+k+q(2n+1)}}{q\pi^{3/2}n!}\left[qB\left(q,\frac{3}{2}\right)\right]^{-(n+\frac{1}{2}+\frac{k}{q})}\Gamma\left(n + \frac{1}{2} + \frac{k}{q}\right)$$
$$\times\left[\sum_{j=0}^{\infty}b^{(j)}\left(n + \frac{1}{2}\right)\left[qB\left(q,\frac{3}{2}\right)\right]^{j/q}\frac{\Gamma(n + \frac{1}{2} + \frac{k}{q} - \frac{j}{q})}{2^j\Gamma(n + \frac{1}{2} + \frac{k}{q})}\right]. \tag{54}$$

We point out that this is the same result that we obtained for odd p except that now all the $E^{(k)}(n+\frac{1}{2})$ are different from zero. We also can read off from equation (54) the general formula of the so-called Bender–Wu coefficients:

$$b_n^{(j)} = b^{(j)}(n + \tfrac{1}{2})2^{-j}[qB(q,\tfrac{3}{2})]^{j/q} \tag{55}$$

and (considering the values of the beta function as a function of q) get a general proof of the fact that they are rational numbers for $p \leqslant 4$ and irrational for $p > 4$. By way of example, we give explicitly the particular result for the sextic anharmonic oscillator up to $j = 1$ in equation (54):

$$E^{(k)}(n+\tfrac{1}{2}) \sim -\frac{2^{4n+5k+5/2}}{\pi^{n+2k+2} n!}\Gamma\left(n+\frac{1}{2}+2k\right) \times\left[1-\frac{\pi^2}{32}\frac{\frac{25}{8}+\frac{221}{25}(n+\frac{1}{2})+\frac{15}{2}(n+\frac{1}{2})^2+\frac{17}{3}(n+\frac{1}{2})^3}{(n+\frac{1}{2}+2k-1)(n+\frac{1}{2}+2k-2)}+\cdots\right]. \tag{56}$$

6. Relation to the JWKB wavefunctions

In this section we discuss briefly the relation between the uniform asymptotic expansions of the previous sections and the JWKB wavefunctions, and restate our matching procedure in terms of loop integrals and Stokes multipliers.

First note that if $\psi_1(z)$ and $\psi_2(z)$ are two linearly independent solutions of the differential equation (4), then the ratio

$$r(z) = \frac{\psi_1(z)}{\psi_2(z)} \tag{57}$$

satisfies

$$\{r, z\} + \frac{2}{h^2}(z^2 - z^p - 2hE) = 0. \tag{58}$$

Since the Schwarzian derivative of the composition of two functions is [22]

$$\{f \circ g, z\} = \{g, z\} + g'(z)^2\{f, g\} \tag{59}$$

it is straightforward to check that

$$S(z) = \frac{h}{2}\ln\left[\frac{\psi_1(z)}{\psi_2(z)}\right] \tag{60}$$

is a solution of the simpler JWKB equation

$$S'(z)^2 = z^2 - z^p - 2hE + \frac{h^2}{2}\{S, z\} \tag{61}$$

that results from the substitution of

$$\psi(z) = [S'(z)]^{-1/2}\exp[S(z)/h] \tag{62}$$

into the Schrödinger equation (4). Therefore we readily identify the $u(z)$-dependent part of the exponential in the definition of $f(\nu)$ (equation (35)) as an asymptotic expansion to twice the action $S_0(z)$ in the classically forbidden region:

$$S_0(z) \sim \frac{h}{2}\ln\left[\frac{u(z)^{\nu}e^{-u(z)^2/(2h)}{}_2F_0(\frac{1}{4}-\frac{\nu}{2},\frac{3}{4}-\frac{\nu}{2};;-hu(z)^{-2})}{u(z)^{-\nu}e^{u(z)^2/(2h)}{}_2F_0(\frac{1}{4}+\frac{\nu}{2},\frac{3}{4}+\frac{\nu}{2};;hu(z)^{-2})}\right] \tag{63}$$

while the $v(z)$ dependent part is also twice an asymptotic expansion of the JWKB action $S_1(z)$ in the classically forbidden region

$$S_1(z) \sim \frac{h}{2}\ln\left[\frac{e^{\frac{2}{3}v(z)^{3/2}/h}{}_2F_0(\frac{1}{6},\frac{5}{6};;\frac{3}{4}hv(z)^{-3/2})}{e^{-\frac{2}{3}v(z)^{3/2}/h}{}_2F_0(\frac{1}{6},\frac{5}{6};;-\frac{3}{4}hv(z)^{-3/2})}\right]. \tag{64}$$

Although this is the most direct way to show the relation between uniform and our JWKB wavefunctions, we would also like to mention that since

$$\frac{d}{dz}\ln\left[\frac{\psi_2(z)}{\psi_1(z)}\right] = \frac{W}{\psi_1(z)\psi_2(z)} \tag{65}$$

where $W \neq 0$ is the constant wronskian of $\psi_1(z)$ and $\psi_2(z)$, therefore

$$\psi_{1,2}(z) = \left(\frac{hW}{2\mathrm{i}}\right)^{1/2} [P(z)]^{-1/2} \exp\left[\pm \frac{\mathrm{i}}{h} \int^z P(t)\,\mathrm{d}t\right] \tag{66}$$

where

$$P(z) = \frac{hW}{2\mathrm{i}} \frac{1}{\psi_1(z)\psi_2(z)} \tag{67}$$

is a solution of

$$P(z)^2 - (z^2 - z^p - 2hE)^2 = h^2 \left[\frac{3}{4}\left(\frac{P'(z)}{P(z)}\right)^2 - \frac{1}{2}\frac{P''(z)}{P(z)}\right]. \tag{68}$$

Equation (66) is the 'phase-integral' form of the JWKB wavefunctions [3] and the relation with our form is simply $S'(z) = \mathrm{i}P(z)$ but, as we will point out later, the main problem solved by the matching procedure is precisely the calculation of the 'integration constant' in passing from $S'(z)$ to $S(z)$. (Calculations of Stokes multipliers within the phase-integral method can be found in [3] for a cluster of two simple transition points, and in [23] for $n \geqslant 2$ simple transition points lying symmetrically on a circle and one transition point of order $n - 2$ in the centre of the circle, although this latter case does not allow for an eigenvalue parameter.)

Our matching procedure consisted of two steps: first, to find the form of the polynomials $E^{(k)}(\nu)$, and second to find the form of the function $f(\nu)$. The quantity ν, which in our formulation appears in the index of the parabolic cylinder function in equation (5), is trivially related to the 'monodromy exponent s of the double turning point' in the geometric terminology of Delabaere *et al* [9]

$$s = -\nu - \tfrac{1}{2} \tag{69}$$

and is determined by the condition

$$\frac{1}{2\pi\mathrm{i}} \oint_\gamma S'(z)\,\mathrm{d}z = \nu \tag{70}$$

where γ is a loop enclosing the origin. If we expand $S'(z)$ as a power series in h

$$S'(z) = \sum_{k=0}^{\infty} S'_k(z) h^k \tag{71}$$

and solve for $S'_k(z)$, the conditions

$$\mathrm{Res}_{z=0} S'_0(z) = \nu \tag{72}$$

$$\mathrm{Res}_{z=0} S'_k(z) = 0 \qquad (k = 1, 2, \ldots) \tag{73}$$

again yield the polynomials $E^{(k)}(\nu)$. This is in essence the calculational algorithm proposed by Delabaere *et al* [9], although they work most of the time with the reverse series (s as a power series in h whose coefficients are polynomials in E) and only at the moment of 'quantization' set $s = n$ and solve for the RSPT series. Furthermore, we can write formally

$$f(\nu) = \frac{(2\pi)^{1/2}}{\Gamma(\nu + \frac{1}{2})} \left(\frac{2}{h}\right)^\nu \exp\left[\frac{2}{h}(S_0(z) - S_1(z))\right] \tag{74}$$

where the prefactor is the Stokes multiplier of [8–11]

$$\frac{(2\pi)^{1/2} x^{-s-1/2}}{\Gamma(-s)} \tag{75}$$

with $x = 2/h$, and the exponent $S_0(z) - S_1(z)$ has to be interpreted as the result of matching two expressions of the same function (the JWKB action in the tunnelling region) with different integration constants, the difference being written as a power series in h. This calculation corresponds to the JWKB tunnelling loop integral 'pinched' by the confluence of two turning points, whose leading contribution is given by the Stokes multiplier and whose higher-order terms are calculated by an 'exact matching method' described in [11] and essentially equivalent to ours, except that again they work with the reverse series, they do not use a Langer–Cherry-type uniform wavefunction but an Olver-type uniform wavefunction to get the leading order, and they get the higher-order terms of a Leray–Gelfand expansion of the loop integral via a Mellin transform introduced by Zinn–Justin (see again [11] and references therein).

The key point of our equations (63) and (64) is that by construction they have the correct integration constants 'built-in,' inherited from the initial conditions imposed to the uniform wavefunctions (8) and (20) respectively, and the matching algorithm is just a series expansion. We finally note that the matching condition (40) can be formally written

$$\Delta\nu = \frac{1}{2\pi \mathrm{i}} \ln\left[1 + \exp\left(\Delta\nu\frac{\partial}{\partial\nu}\right) f(\nu)\right]_{\nu=n+\frac{1}{2}} \tag{76}$$

which is an instance of an 'alien derivative' equation in the terminology of [8–11].

7. Summary

In this paper we have built and matched uniform asymptotic expansions for the wavefunction around the single and double turning points in the anharmonic oscillators with potentials $V(x) = \frac{1}{2}x^2 - gx^p$. The wavefunction anchored at the double turning point at the origin is taken as a Langer–Cherry expansion based on a parabolic cylinder function with an unspecified index $\nu - \frac{1}{2}$, while the wavefunction anchored at the simple outer turning point is a similar expansion based on an Airy function. Imposing appropriate initial conditions to these uniform expansions yields a series that is formally the RSPT series, but with $n + \frac{1}{2}$ replaced by the as yet unspecified parameter ν, and the matching condition is precisely an equation to determine ν—an idea that can be traced to [19].

The main advantage of this approach is that it encodes the asymptotic solution of the eigenvalue problem for the resonances in just two steps: the very simple calculation of the RSPT power series as a polynomial in $n + \frac{1}{2}$, and the more complicated but still algorithmic determination of the polynomials appearing in the matching function $f(\nu)$.

In fact, matching in a suitable sector of the complex g plane (which depends on the parity of the perturbation) yields $\nu = n + \frac{1}{2}$ and the Borel-summable RSPT series, while in the next Riemann sheet yields $\nu = n + \frac{1}{2} + \Delta\nu$ with an explicit equation (40) or (50) for the determination of $\Delta\nu$.

We have solved the matching conditions to first exponentially small order, and given a unified derivation of the asymptotic behaviour of the imaginary part of the resonances for all these oscillators and, via dispersion relations, the corresponding asymptotic behaviour of the RSPT coefficients.

Finally, we have discussed the relation between the uniform asymptotic and the JWKB wavefunctions, and between our matching procedure and the matching in terms of Stokes multipliers and JWKB loop integrals.

Particular cases (quartic [24], cubic [25] and sextic [26]) of the RSPT coefficients written as polynomials in n or $n + \frac{1}{2}$ had been discussed in different contexts, as well as the formal relation between semiclassical methods and perturbation theory [26], but the analytic role of the RSPT expansion (as a function of the parameter ν) in the matching process, introduced in [19] and applied in this paper, had not been clarified. As a future line of work we would like also to mention the calculation and interpretation of higher exponentially small-order solutions to the matching condition, and the related study of other complex turning points which play a role in the connection path.

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